Algebraic Geometry Lecture 14 – Intersections in Projective Space Part II

We need to recap/know the following ring theory.

Def$^{\mathbf{n}}$. A *multiplicative system* in a ring S is a subset A containing 1 and closed under multiplication. The localisation $A^{-1}S$ is defined to be the ring formed by equivalence classes of fractions s/a with $s \in S$, $a \in A$, where

$$s/a \sim s'/a' \quad \Leftrightarrow \quad \text{there exists } a'' \in A \text{ such that } a''(a's - as') = 0.$$

If p is a prime ideal in S then $A = S\backslash p$ is a multiplicative system. The localisation $A^{-1}S$ is then denoted S_p.

Example Consider the ring $\mathbb{Z}$. For any prime number p, $A = \mathbb{Z} \setminus p\mathbb{Z}$ is a multiplicative system, for if $ab \in p\mathbb{Z}$ then either a or b must be in $p\mathbb{Z}$, so if $a, b \in A$ then $ab \in A$. The localisation is then

$$\mathbb{Z}_{p\mathbb{Z}} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus p\mathbb{Z}\} \Big/ \sim$$

where $a/b \sim a'/b'$ if and only if there is a $c \in \mathbb{Z} \setminus p\mathbb{Z}$ such that

$$c(b'a - ba') = 0.$$

But $c \neq 0$ so this just means $b'a - ba' = 0$, or simply $a/b = a'/b'$, noting that neither b nor b' can be zero. So we have

$$\mathbb{Z}_{p\mathbb{Z}} = \left\{ \frac{a}{b} : p \nmid b \right\}$$
$$(= \mathbb{Q} \cap \mathbb{Z}_p).$$

The Hilbert Polynomial ctd.

Recall we're trying to generalise Bézout's theorem on how many times two curves intersect counting multiplicities. By the end of this lecture we will be able to state how many times a variety intersects a hypersurface, but we still need to define the degree of a polynomial and how to count the multiplicity of an intersection. First we need to introduce graded rings and modules.

⌜ Recall: A *module* over a ring S, or S-module, is an abelian group M such that we can multiply by elements of S, so for all $s, s_1, s_2 \in S$, $m, m_1, m_2 \in M$:

- $s_1(s_2 m) = (s_1 s_2)m$
- $(s_1 + s_2)m = s_1 m + s_2 m$
- $s(m_1 + m_2) = sm_1 + sm_2$.

We say the module is finitely generated if there is a finite set of elements in M, say $m_1, \ldots, m_r$ such that any element of M can be written as a linear combination of these r elements over the ring S.

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Def$^{\mathbf{n}}$. A *graded ring* is a ring S together with a decomposition as a direct sum

$$S = \bigoplus_{d \geqslant 0} S_d$$

with S_d abelian groups, such that if $a \in S_d$ and $b \in S_e$ then $ab \in S_{d+e}$.

Def$^{\mathbf{n}}$. (i) An element of S_d is called a *homogeneous element of degree d*.

(ii) An ideal $\mathcal{A} \subseteq S$ is a *homogeneous ideal* if

$$\mathcal{A} = \bigoplus_{d \geqslant 0} (\mathcal{A} \cap S_d).$$

Def$^{\mathbf{n}}$. If S is a graded ring then a *graded S-module* is an S-module M with a decomposition

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

such that if $s \in S_d$ and $m \in M_e$ then $sm \in M_{d+e}$.

Given an algebraic set Y we will be taking S to be $k[x_0, \ldots, x_n]$, and M to be $k[Y] = S/I(Y)$, the homogeneous coordinate ring of Y. We have

$$S = \bigoplus_{d \geqslant 0} S_d$$

where S_d are the homogeneous polynomials in S of degree d, and

$$M = \bigoplus_{d \in \mathbb{Z}} S_d / I_d(Y),$$

where $I_d(Y) = I(Y) \cap S_d$.

Now for a few more definitions.

Def$^{\mathbf{n}}$. For any graded S-module M and any integer ℓ the *twisted* module $M(\ell)$ is formed by shifting the decomposition of M ℓ places to the left. So if

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

then

$$M(\ell) = \bigoplus_{d \in \mathbb{Z}} M_{d+\ell}.$$

Def$^{\mathbf{n}}$. If M is a graded S-module then the *annihilator* of M is

$$\operatorname{Ann} M = \{s \in S \mid sM = 0\}.$$

It is a homogeneous ideal in S.

Def$^{\mathbf{n}}$. Let M be a graded module over $S = k[x_0, \dots, x_n]$. The *Hilbert function* of M is

$$\varphi_M(\ell) = \dim_k M_\ell$$

for all $\ell \in \mathbb{Z}$.

Theorem (Hilbert–Serre)**.** *Let M be a finitely generated graded S-module where $S = k[x_0, \dots, x_n]$. Then there exists a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that*

$$\varphi_M(\ell) = P_M(\ell)$$

for all sufficiently large integers ℓ. Furthermore,

$$\deg P_M(z) = \dim V(\operatorname{Ann} M).$$

Def$^{\mathbf{n}}$. The polynomial P_M in the above theorem is the *Hilbert polynomial* of M.

Def$^{\mathbf{n}}$. If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r, we define the *Hilbert polynomial of* Y to be the Hilbert polynomial of its homogeneous coordinate ring $k[Y]$, denoted P_Y. By the Hilbert-Serre theorem it has degree r. Define the *degree of* Y to be $r!$ times the leading coefficient of P_Y.

Example Let $Y \subset \mathbb{P}^2$ be the variety

$$Y : z - x = 0.$$

Note that this is a projective line, so has dimension 1. Our graded ring is the ring of polynomials which we're denoting $S = k[x, y, z]$. It has decomposition

$$S = \bigoplus_{d \geqslant 0} k[x, y, z]_d$$

where $k[x, y, z]_d$ are the homogeneous polynomials of degree d. Our graded module over S is the homogeneous coordinate ring $k[Y] = S/I(Y)$ which has decomposition

$$M = \bigoplus_{d \in \mathbb{Z}} k[x, y, z]_d/(z - x)_d = \bigoplus_{d \in \mathbb{Z}} k[x, y]_d.$$

So the Hilbert function is given by

$$\begin{aligned}
\varphi_M(\ell) &= \dim_k M_\ell \\
&= \dim_k k[x, y]_\ell \\
&= \ell + 1.
\end{aligned}$$

So the Hilbert polynomial is just $P_M(z) = z + 1$, and thus $\deg(Y) = 1$.

Def$^{\mathbf{n}}$. Let M be a module over the ring S. A filtration of M is an ascending chain of submodules of

$$0 = M^0 \subset M^1 \subset \ldots \subset M^n = M.$$

We define the *length* of the filtration to be n. The length of M is the maximum length of any of its filtrations.

We are now in a position to start defining the multiplicity of an intersection. The definition when we get to it would seem to be not well defined, but the following proposition takes care of that.

Proposition 1. *Let M be a finitely generated graded module over a noetherian graded ring S (noetherian means all prime ideals are finitely generated). Then there exists a filtration*

$$0 = M^0 \subset M^1 \subset \ldots \subset M^r = M$$

by graded submodules such that for each i,

$$M^i/M^{i-1} \cong (S/p_i)(\ell_i)$$

for a homogeneous prime ideal p_i of S and some integer ℓ_i. The filtration is not necessarily unique, but to each such filtration we get a collection of (not necessarily distinct) prime ideals $\{p_1, \ldots, p_r\}$. And while the filtration is not unique, for any such filtration the collection of prime ideals is the same.

With this in the bag we can now define:

Def$^{\mathbf{n}}$. If p is a minimal prime ideal of a graded S-module M then the multiplicity of M at p, denoted $\mu_p(M)$, is defined to be the number of times that p appears in any filtration as above.

And now at last we can define the following.

Def$^{\mathbf{n}}$. Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let H be a hypersurface not containing Y, i.e. $Y \nsubseteq H$. Then by the projective dimension theorem $Y \cap Z = Z_1 \cup \ldots \cap Z_s$, where each Z_j is a variety of dimension $r - 1$. Let $p_j = I(Z_j)$ be the homogeneous prime ideal of Z_j. We define the *intersection multiplicity* of Y and H along Z_j to be

$$i(Y, H; Z_j) = \mu_{p_j}\left(S/(I(Y) + I(H))\right).$$

Theorem. *Let $Y \subseteq \mathbb{P}^n$ be a variety of dimension $\geqslant 1$, and let H be a hypersurface not containing Y. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then*

$$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = (\deg Y)(\deg H).$$

As you may have noticed working out the Hilbert polynomials is a big challenge in most cases, though Groebner bases can help with it. But we can work out the Hilbert polynomial of a point and so give a proof of Bézout's theorem.

Corollary (Bézout's theorem)**.** *Let Y, Z be distinct curves in $\mathbb{P}^2$ having degrees d, e respectively. Let $Y \cap Z = \{P_1, \ldots, P_s\}$. Then*

$$\sum_{j=1}^{s} i(Y, Z; P_j) = de.$$

Proof. By the projective dimension theorem the P_j's must be points and it's an easy exercise to show that the Hilbert polynomial of a point is just 1, and so $\deg P_j = 1$ for each j, and so the result follows from the theorem. $\square$